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KENNETH J. ARROW

TECHNICAL REPORT NO. 142
March 31, 1966

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FOR
OFFICE OF NAVAL RESEARCH

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
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1. History of the Concept

There are perhaps two basic, though incompletely separable, aspects of the notion of general equilibrium as it has been used in economics: (1) the simple notion of determinateness, that the relations which describe the economic system must form a system sufficiently complete to determine the values of its variables, and (2) the more specific notion that each relation represents a balance of forces. The last usually, though not always, is taken to mean that a violation of any one relation sets in motion forces tending to restore it (as will be seen below, this is not the same as the stability of the entire system). In a sense, therefore, almost any attempt to give a theory of the whole economic system implies the acceptance of the first part of the equilibrium notion; and Adam Smith's "invisible hand" is a poetic expression of the most fundamental of economic balance relations, the equalization of rates of return, as enforced by the tendency of factors to move from low to high returns.

The notion of equilibrium ("equal weight," referring to the condition for balancing a lever pivoted at its center) had been familiar in mechanics long before 1776, and with it the notion that the effects of a force may annihilate it (e.g., water finding its own level), but there is no

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obvious evidence that Smith drew his ideas from any analogy with mechanics. Whatever the source of the concept, the notion that, through the workings of an entire system, effects may be very different from and even opposed to intentions is surely the most important intellectual contribution that economic thought has made to the general understanding of social processes.

It can thus be maintained that Smith was a creator of general equilibrium theory, though the coherence and consistency of his work can be questioned. A fortiori, later systematic expositors of the classical system, such as Ricardo, Mill, and Marx, in whose work some of Smith's logical gaps were filled, can all be regarded as early expositors of general equilibrium theory. Marx, indeed, in his scheme of simple reproduction read in combination with his development of relative price theory in Volumes I and III, has come in some ways closer in form to modern theory than any other classical economist, though of course everything is confused by his attempt to maintain simultaneously a pure labor theory of value and an equalization of rates of return on capital.

The view that the classical economists had a form of general equilibrium principle is further bolstered by modern reconstructions. D. J. Schouten has indeed presented systematic complete models which are intended to represent the systems of different economists, and Samuelson has done the same for Ricardo and for Marx.

There is, however, a very important sense in which none of the classical economists had a true general equilibrium theory: none had an explicit role for demand conditions. No doubt the more systematic thinkers among them, J. S. Mill and Cournot most particularly, gave verbal homage to the role of demand and the influence of prices on it, but there was no

genuine integration of demand with the essentially supply-oriented nature of classical theory. The neglect of demand was facilitated by the special simplifying assumptions made abc t supply. A general equilibrium theory, from the modern point of view, is a theory about both the quantities and the prices of all economic magnitudes. However, the classical authors found that prices appeared to be determined by a system of relations not involving quantities, derived from the zero-profit condition. This is clear enough with fixed production coefficients and a single primary factor, labor, as in Smith's famous exchange of deer and beavers; and it was the great accomplishment of Malthus and Ricardo to show that land could be brought into the system. If finally Malthusian assumptions about population implied that the supply price of labor was fixed in terms of goods, then even the price of capital could be determined (though the presence of capital as a productive factor and recipient of rewards was clearly an embarrassment to the classical authors, as it remains to some extent today).

Thus, in a certain definite sense the classicists had no true theory of resource allocation since the influence of prices on quantities was not studied, and the reciprocal influence denied. But the classical theory could not survive the logical problem of explaining relative wages of heterogeneous types of labor nor the empirical problem of accounting for wages which were rising steadily above the subsistence level. It is in this context that the neoclassical theories emerged, with all primary resources having the role that land alone had before.

(In all fairness to the classical writers, it should be remarked that the theory of foreign trade, especially in the form given to it by J. S. Mill, was a genuine general equilibrium theory. But of course the assumptions made, particularly factor immobility, were very restrictive.)

The full recognition of the general equilibrium concept can unmistakably, therefore, be attributed to Léon Walras, though many of the elements of the neoclassical synthesis had been worked out independently by W. Stanley Jevons and by Carl Menger. The economic system was made up of households and firms. Each household owns a set of resources, commodities useful in production or consumption, including different kinds of labor. For any given set of prices, a household then has an income from the sale of its resources, and with this income it can choose among all alternative bundles of consumers' goods whose cost, at the given prices, does not exceed the household's income. Thus, the demand by households for any consumers' good is a function of the prices of both consumers' goods and resources. The firms were (at least in the earlier versions) assumed to be operating under fixed coefficients. Then the demand for consumers' goods determined the demand for resources; and the combined assumptions of fixed coefficients and zero profits for a competitive system implied relations between the prices of consumers' goods and of resources. An equilibrium set of prices, then, was a set such that supply and demand were equated on each market; under the assumption of fixed coefficients of production, or more generally of constant returns to scale, this amounted to equating supply and demand on the resource markets, with prices constrained to satisfy the zero-profit conditions for firms. Subsequent work of Walras, J. B. Clark, Wicksteed and others generalized the assumptions about production to include alternative methods of production, as expressed in a production function. Then the prices of resources were determined by marginal productivity considerations.

That there existed an equilibrium set of prices was argued from the equality of the number of prices to be determined with the number of

equations expressing the equality of supply and demand on various markets. In this counting, Walras recognized that there were two offsetting complications: (1) Only relative prices affected the behavior of households and firms; hence, the system of equations only had $n-1$ variables, a point which Walras expressed by selecting one commodity to serve as <u>numéraire</u>, with the prices of all commodities being measured relative to it. (2) The budgetary balance of each household between income and the value of consumption and the zero-profit condition for firms together imply Walras' Law, that the market value of supply equal that of demand for any set of prices, not merely the equilibrium set; hence, the supply-demand relations are not independent. If supply equals demand on $n-1$ markets, then the equality must hold on the n^{th}.

Walras wished to go further and discussed the stability of equilibrium essentially for the first time (that is, apart from some brief discussions by Mill in the context of foreign trade), in his famous but rather clumsy theory of <u>tâtonnements</u> (literally, "gropings," or "tentative proceedings"). He starts by supposing a set of prices set arbitrarily; then supply may exceed demand on some markets and fall below on others (unless the initial set is in fact the equilibrium set, there must be at least one case of each, by Walras' Law). Suppose the markets considered in some definite order. On the first market, adjust the price so that supply and demand are equal, given all other prices; this will normally require raising the price if demand initially exceeded supply, decreasing it in the opposite case. The change in the first price will, of course, change supply and demand on all other markets. Repeat the process with the second and subsequent markets. At the end of one round, the last market will be in equilibrium, but none of the others need be since the adjustments on

subsequent markets will destroy the equilibrium achieved on any one. However, Walras argues, the supply and demand functions for any given commodity will be more affected by the changes in its own price than by the changes in other prices; hence, after one round the markets should be more nearly in equilibrium than they were to begin with, and with successive rounds the supplies and demands on all markets will tend to equality.

It seems clear that Walras did not literally suppose that the markets came into equilibrium in some definite order. Rather, the story was a convenient way of showing how the market system could in fact solve the system of equilibrium relations. The dynamic system, more properly expressed, asserted that the price on any market rose when demand exceeded supply, and fell in the opposite case; the price changes on the different markets were to be thought of as occurring simultaneously.

Finally, Walras had a still higher aim for general equilibrium analysis, to study what is now called <u>comparative statics</u>, by which is meant the laws by which the equilibrium prices and quantities varied with the underlying data (resources, production conditions, or utility functions). But little was actually done in this direction.

Important contributions were made by Walras' contemporaries, Edgeworth, Pareto, and Irving Fisher. One perhaps calls for special mention since it has again become the subject of significant research, the "<u>contract curve</u>" (Edgeworth, 1881), known in modern terminology as the <u>core</u> (see Section 3 below, and Debreu and Scarf).

But the next truly major advances did not come until the 1930's. There were two distinct streams of thought, one beginning in German-language literature and dealing primarily with the existence and

uniqueness of equilibrium, the other primarily in English and dealing with stability and comparative statics. The former started with a thorough examination of Cassel's simplification of Walras' system, an interesting case of work which had no significance in itself and yet whose study turned out to be extraordinarily fruitful. Cassel assumed two kinds of goods: commodities which entered into the demand functions of consumers, and factors which were used to produce commodities (intermediate goods were not considered). Each commodity was produced from factors with constant input-output coefficients. Factor supplies were supposed totally inelastic. Let a_{ij} be the amount of factor i used in the production of one unit of commodity j, x_j the total output of commodity j, v_i the total initial supply of factor i, p_j the price of commodity j, and r_i the price of factor i. Then the condition that demand equal supply for all factors reads

(1) $$\sum_j a_{ij}x_j = v_i ,$$

while the condition that each commodity be produced with zero profits reads

(2) $$\sum_i a_{ij}r_i = p_j .$$

The system is completed by the equations relating the demand for commodities to their prices. In total, there are as many equations as unknowns. But three virtually simultaneous papers in 1932 (Zeuthen, 1932-3; Neisser, 1932, and von Stackelberg, 1932-3) showed in different ways that the problem of existence of meaningful equilibrium was deeper than equality of equations and unknowns. Neisser noted that even with perfectly plausible values of the input-output coefficients a_{ij}, the prices or quantities

which satisfied (1) and (2) might well be negative. Von Stackelberg noted that (1) constituted a complete system of equations in the outputs x_j, since the factor supplies, v_i, were data, but nothing had been assumed about the numbers of distinct factors or distinct commodities. If, in particular, the number of commodities exceeded that of factors, equations (1) would in general have no solutions.

Zeuthen reconsidered the meaning of equations (1). He noted that economists, at least since Carl Menger, had recognized that some factors (e.g., air) were so abundant that there would be no price charged for them. These would not enter into the list of factors in Cassel's system. But, Zeuthen argued, the division of factors into free and scarce should not be taken as given a priori. Hence, all that can be said is that the usage of a factor should not exceed its supply, but if it falls short, then the factor is free. In symbols, (1) is replaced by

$$(1') \qquad \sum_j a_{ij} x_j \leq v_i; \text{ if the strict inequality holds, then } r_i = 0.$$

To a later generation of economists to whom linear programming and its generalizations are familiar, the crucial significance of this step will need no elaboration; equalities are replaced by inequalities and the vital notion of the complementary slackness of quantities and prices introduced.

Independently of Zeuthen, the Viennese banker and amateur economist, R. Schlesinger, came to the same conclusion. But he went much further and intuitively grasped the essential point, that replacement of equalities by inequalities also resolved the problems raised by Neisser and von Stackelberg. Schlesinger realized the mathematical complexity of a rigorous treatment and, at his request, Oskar Morgenstern put him in touch with a

young mathematician, Abraham Wald. The result was the first rigorous analysis of general competitive equilibrium. In a series of papers (see Wald, 1936, for a summary), he demonstrated the existence of competitive equilibrium in a series of alternative models including the Cassel model and a model of pure exchange. Competitive equilibrium was defined in the Zeuthen sense, and indeed the essential role of that definition in the justification of existence is made clear in the mathematics. Wald also initiated the study of uniqueness. Indeed, both of his alternative sufficient conditions have since become major themes of the literature: (1) that the weak axiom of revealed preference hold for the demand functions of the entire market, or (2) that all commodities be gross substitutes (see definition below).

Wald's papers were of forbidding mathematical depth, not only in the use of sophisticated tools but also in the complexity of the argument. As they gradually came to be known among mathematical economists, they probably served as much to inhibit further research by their difficulty as to stimulate it. Help finally came from development of a related line of research, John von Neumann's theory of games (first basic paper published in 1928; see von Neumann and Morgenstern, 1944). The historical relation between game theory and economic equilibrium theory is paradoxical. In principle, game theory is a very general notion of equilibrium which should either replace the principle of competitive equilibrium or include it as a special case. In fact, while game theory has turned out to be extraordinarily stimulating to equilibrium theory, it has been through the use of mathematical tools developed in the former and usedin the latter with entirely different interpretations. It was von Neumann himself who made the

first such application in his celebrated paper on balanced economic growth (von Neumann, 1937). In this model there were no demand functions, only production. The markets had to be in equilibrium in the Zeuthen sense. But beyond this there was equilibrium in a second sense which may be termed stationary equilibrium (see Section 7 below). To prove the existence of equilibrium, von Neumann demonstrated that a certain ratio of bilinear forms had a saddle-point, a generalization of the theorem which showed the existence of equilibrium in two-person zero-sum games. But in game theory the variables of the problem were probabilities (of choosing alternative strategies), while in the application to equilibrium theory one set of variables was prices and the other the levels at which productive activities were carried on.

Von Neumann deduced his saddle-point theorem from a generalization of Brouwer's fixed-point theorem, a famous proposition in the branch of mathematics known as topology. A simplified version of von Neumann's theorem was presented a few years later by the mathematician, Shizuo Kakutani, and Kakutani's theorem has been the basic tool in virtually all subsequent work. With these foundations, plus the influence of the rapid development of linear programming on both the mathematical (again closely related to saddle-point theorems) and economic sides (the work of George B. Dantzig, Albert W. Tucker, Harold W. Kuhn, Tjalling C. Koopmans, and others, collected for the most part in an influential volume (Koopmans, 1951)) and the work of John F. Nash, Jr., it was perceived independently by a number of scholars that existence theorems of greater simplicity and generality than Wald's were now possible. The first papers were those of Lionel McKenzie (1954) and Kenneth J. Arrow and Gerard Debreu (1954). Subsequent

developments were due to Hukukane Nikaidô, Hirofumi Uzawa, Debreu, and McKenzie. The most complete systematic account of the existence conditions is in Debreu (1959); the most general version of the theorem is in Debreu (1962).

Independently of this development of the existence conditions for equilibrium, the Anglo-American literature contained an intensive study of the comparative statics and stability of general competitive equilibrium. Historically, it was closely related to analyses of the second-order conditions for maximization of profits by firms and of utility by consumers; the most important contributors were John R. Hicks, Harold Hotelling, Paul Samuelson, and R. G. D. Allen. In particular, Hicks introduced the argument that the stability of equilibrium carried with it some implications for the shapes of the supply and demand functions in the neighborhood of equilibrium; hence, the effects of small shifts in any one behavior relation may be predicted, at least as to sign. Hicks's definition of stability has been replaced in subsequent work by Samuelson's; however, he showed that (locally) stability in his sense was equivalent to a condition which has played a considerable role in subsequent research. Let X_i be the excess demand (demand less supply) for the i^{th} commodity; it is in general a function of $p_1,\ldots,p_n$, the prices of all n commodities. Then Hicks's definition of stability was equivalent to the condition that the principal minors of the matrix whose elements were $\partial X_i/\partial p_j$ had determinants which were positive or negative according as the number of rows or columns included was even or odd. Such matrices will be referred to as Hicksian. The laws of comparative statics which Hicks sought to derive have remained the only ones valid, though different sufficient conditions for their validity are now accepted.

Samuelson formulated the presently accepted definition of stability. It must be based, he argued, on an explicit dynamic model concerning the behavior of prices when the system is out of equilibrium. He formalized the implicit assumption of Walras and most of his successors: the price of each commodity increased at a rate proportional to excess demand for that commodity. This assumption defined a system of differential equations: if every path satisfying the system and starting sufficiently close to equilibrium converged to it, then the system was stable. Samuelson was able to demonstrate that Hicks's definition was neither necessary nor sufficient for his, and that the economic system was stable if the income effects on consumption were sufficiently small. He enunciated a general Correspondence Principle, that all meaningful theorems were derived either from the second-order conditions on maximization of profits by firms or of utility by consumers or from the assumption that the observed equilibrium was stable.

The current period of work in comparative statics and stability dates from the work of Mosak (1944) and Metzler (1945). The emphasis has tended to be a little different from Samuelson's Correspondence Principle; rather, the tendency has been to formulate hypotheses about the excess demand function which imply both stability and certain results in comparative statics.

2. The Existence of Competitive Equilibrium

Consider a system with n commodities, with prices $p_1, \ldots, p_n$, respectively. Let us first suppose that at each set of prices, each economic agent (firm or household) has a single chosen demand or supply. If supplies are treated as negative demands, then for each commodity the

net total excess demand (excess of demand over supply) by all economic agents is obtained by summing the excess demands for the individual agents, and is a function of all prices; let $X_i(p_1,\ldots,p_n)$ be the excess demand function for the i^{th} commodity. At an equilibrium, excess demand cannot be positive since there is no way of meeting it. Further, if the excess demand is negative (i.e., there is an excess of supply over demand), the good is free and should have a zero price. Formally, a set of non-negative prices $\bar{p}_1,\ldots,\bar{p}_n$ constitutes an <u>equilibrium</u> if

(1) $$X_i(\bar{p}_1,\ldots,\bar{p}_n) \leqq 0 \quad \text{for all} \quad i,$$

(2) $$\bar{p}_i = 0 \quad \text{for all} \quad i \quad \text{such that}$$
$$X_i(\bar{p}_1,\ldots,\bar{p}_n) < 0.$$

With this definition, the following assumptions are sufficient for the existence of equilibrium:

(H) The functions $X_i(p_1,\ldots,p_n)$ are (positively) homogeneous of degree zero (i.e., $X_i(\lambda p_1,\ldots,\lambda p_n) = X_i(p_1,\ldots,p_n)$ for all $\lambda > 0$ and all $p_1,\ldots,p_n$).

(W) $\sum_i p_i X_i(p_1,\ldots,p_n) = 0$ for all sets of prices.

(C) The functions $X_i(p_1,\ldots,p_n)$ are continuous.

(B) The functions $X_i(p_1,\ldots,p_n)$ are bounded from below (i.e., supply is always limited.

Assumption (H) is standard in consumers' demand theory; (W) is Walras' Law referred to above. Assumption (C) is the type usually made in any applied work although, as will be seen later, the hypothesis is

closely related to assumptions of convexity of preferences and production. Assumption (B) is trivially valid in a pure exchange economy since each individual has a finite stock of goods to trade. In an economy where production takes place, the matter is less clear. At an arbitrarily given set of prices, a producer may find it profitable to offer an infinite supply; the realization of his plans will, of course, require him to demand at the same time an infinite amount of some factor of production. Such situations are of course incompatible with equilibrium, but since the existence of equilibrium is itself in question here, the analysis is necessarily delicate.

The current proofs that the assumptions listed above imply the existence of competitive equilibrium require the use of Brouwer's fixed-point theorem, a mathematical theorem which asserts that a continuous transformation of a triangle or similar figure in higher-dimensional spaces into itself must leave at least one point unaltered. The argument may be sketched as follows: From (H), an equilibrium is unaltered if all prices are multiplied by the same positive constant; hence, without loss of generality we can assume that the sum of the prices is one. The set of all price vectors with non-negative components summing to one is clearly a generalized triangle (technically called a simplex). For each set of prices, compute the excess demands (positive or negative) and then form a new price vector in which those components with positive excess demands are increased and the others decreased (but not below zero). These new prices are then adjusted so that the sum is again one. This process defines a continuous transformation of the simplex into itself and thus has a fixed point, a price vector which remains unaltered under the adjustment process. It is easy to see that this price vector must be an equilibrium.

The point of view just sketched is not sufficiently general for most purposes. We have already seen that the boundedness assumption appears artificial in the case where production is possible. Closely related to this is a second issue; the assumption that supplies and demands are single-valued appears unduly restrictive. Consider the simplest case of production: one input, one output which is proportional to the input. The behavior of the profit-maximizing entrepreneur depends on the ratio of the output price to the input price. If the price ratio is less than the output-input ratio, then the firm will lose money if it engages in any production; hence, the profit-maximizing point is zero output and zero input, which is indeed single-valued. If the two ratios are equal, however, all output levels make zero profit; hence, the profit-maximizing entrepreneur is indifferent among them, and the supply function of the output and the demand function for the input must be taken to be multi-valued. If the price ratio is higher than the output-input ratio, then the entrepreneur will make increasing profits as he increases the scale of activity. There is *no* finite level which could be described as profit-maximizing.

To state a general definition of competitive equilibrium more precisely, the following model can be formulated: There are presumed to exist a set of households and a set of firms; all production is carried on in the latter. Each household has a collection of initial assets (here assumed to include the ability to supply different kinds of labor) and also a claim to a given portion (possibly zero, of course) of the profits of each firm; it is assumed that for each firm there are claims for exactly the entire profits (these claims are interpretable as equities or partnership shares). For given prices and given production decisions of the firms,

the profits of the firms and the values of each individual's initial assets are determined, and hence so is the individual's total income. The commodity bundles available are those whose value, at the given prices, does not exceed income and whose individual components are non-negative (or satisfy some still stronger condition independent of prices). It is further assumed that the household can express preferences among commodity bundles and that these preferences have suitable continuity properties. Then the aim of the household is taken to be selection of the most preferred bundle among those available.

The behavior of the firms is more simply described. Each firm has available to it a set of possible production bundles; conventionally, the components are taken to be positive for outputs and negative for inputs. For a fixed set of prices, the profits for each possible production bundle are determined; then the firm chooses the (or a) bundle which maximizes profits. Notice that the profit-maximizing bundle need not be unique. Indeed, under constant returns to scale it is never unique unless the firm's best policy is to shut down completely. (Under constant returns, if any bundle is possible, the bundle obtained by doubling all components, inputs and outputs alike, is also possible. Then if any bundle makes positive profits, doubling the bundle will double the profits so there can be no profit-maximizing bundle. The existence of a profit-maximizing bundle thus entails that maximum profits be non-positive. Since zero profits are always possible by zero activity level, we must have zero profits at the maximum. Either profits are negative for all non-zero bundles, in which case shutting down is the unique optimal policy, or profits are zero for some non-zero bundle, in which case any non-negative

multiple also achieves zero and therefore maximum profits.)

A <u>competitive equilibrium</u>, then, is a designation of non-negative prices for all commodities, of a bundle for consumption for each household, and of a production bundle for each firm satisfying the following conditions:

(a) for each household, the designated bundle maximizes utility among all available bundles;

(b) for each firm, the designated bundle maximizes profit among all technically possible bundles;

(c) for each commodity, the total consumed by all households does not exceed the total initially available plus the net total produced by all firms ("net" here means that input uses by some firms are subtracted from outputs of others);

(d) for those commodities for which total consumed is strictly less than total initially available plus total produced, the price is zero.

The following assumptions are sufficient to insure the existence of competitive equilibrium:

(I) The preference ordering of each household is continuous (a strict preference between two bundles continues to hold if either is slightly altered), admits of no saturation (for each bundle, there is another preferred to it), and is convex (if a bundle is varied along a line segment in the commodity space, one of the endpoints is least preferred).

(II) The set of possible production bundles for each firm is convex (any weighted combination of two possible production bundles is possible) and closed (any bundle that can be approximated by possible bundles is

itself possible); further, it is always possible to produce no outputs and use no inputs.

(III) No production bundle possible to society as a whole (a bundle is possible for society as a whole if it is the algebraic sum of production bundles, one chosen among those possible for each firm) can contain outputs but not inputs; there is at least one bundle possible for society which produces a positive net output of all commodities not possessed initially by any household.

(IV) The economy is irreducible (a concept due to Gale and McKenzie) in the sense that no matter how the households are divided into two parts, an increase in those initial assets held by the members of one group can be used to make feasible an allocation which will make no one worse off and at least one individual in the second group better off.

It is perhaps interesting to observe that "atomistic" assumptions concerning individual households and firms are not sufficient to establish the existence of equilibrium; "global" assumptions III and IV are also needed (though they are surely unexceptionable). Thus, a limit is set to the tendency implicit in price theory, particularly in its mathematical versions, to deduce all properties of aggregate behavior from assumptions about individual economic agents.

The hypotheses of convexity in household preferences and in production are the empirically most vulnerable parts of the above assumptions. In production, convexity excludes indivisibilities or increasing returns to scale. In consumption, convexity excludes cases in which mixed bundles are inferior to extremes; for example, in the very short run a mixture of

whisky and gin is regarded by many as inferior to either alone, or living part time in each of two distant cities may be inferior to living in either alone. It is of interest to know how far these assumptions may be relaxed.

Convexity does play an essential role in the proof. This may be illustrated by considering the simplest case, one input and one output proportional to it. As noted earlier, there will be one ratio of output to input prices at which the entrepreneur will be indifferent among all levels of output. If the supply of the input is given, then the equilibrium levels of input and output, as well as price, are determined. Now suppose that production is possible only at integer-valued levels of input so that the production possibility set is not convex. If the supply of the input is not an integer, there is no way of equating demand and supply for it. It should be noted, though, that the input (and output) level can be so chosen that the difference between supply and demand does not exceed one-half. In effect, convexity insures that supply and demand are, in a suitable sense, continuous and thus can be adjusted to varying levels of initial assets.

The assumption of convexity cannot be dispensed with in general theorems concerning the existence of equilibrium strictly defined. However, a line of thought begun by M. J. Farrell and developed by J. Rothenberg and R. J. Aumann suggests that the gap between supply and demand does not tend to increase with the size of the economy. Thus, if each agent (household or firm) is small compared with the total economy, then by suitable choice of prices and of consumption and production bundles, the discrepancy between supply and demand can be made small relative to the economy. Each household is certainly small relative to the economy, so that non-convexities in individual preferences have no significant effect on the

existence of equilibrium. However, sufficiently rapidly increasing returns to scale may mean that a competitive profit-maximizing firm will be large at a given set of prices, and hence there may be a real possibility that equilibrium not exist.

3. Optimality and the Core

Though the view that competitive equilibria have some special optimality properties is at least as old as Adam Smith's invisible hand, a clarification of the relation is fairly recent. Since the subject belongs to the field of Welfare Economics (q.v.), only a brief statement is given here. An allocation (designation of bundles for all households and all firms) is <u>feasible</u> if each bundle is possible for the corresponding agent and if, in the aggregate, the net output of each commodity (including quantities initially available) is at least as great as the demand by consumers. Each allocation then determines the utility level of the consumption of each household. One allocation is <u>dominated</u> by a second if the latter is feasible and if each individual has a higher utility under the second than under the first (more frequently, in the literature, the condition is put in the more complicated form of having each individual at least as well off and one individual better off, but the difference is trifling). Then an allocation is said to be optimal if it is feasible but not dominated by any other (a definition due to Pareto).

There are two theorems relating competitive equilibrium and optimal allocations, concerning sufficiency and necessity; the two have not always been distinguished in the literature. They are stated here without some minor qualifications.

<u>Sufficiency</u>. Any competitive equilibrium is necessarily optimal.

Necessity. Given any optimal allocation, there is some assignment of society's initial assets among individuals such that the optimal allocation is a competitive equilibrium corresponding to that distribution, providing that the assumptions of Section 2, which insure the existence of equilibrium, hold.

It is useful to note that the sufficiency theorem is valid even if the assumptions of Section 2 do not hold.

The concept of optimality is defined without regard to a price system or any prescribed set of markets. The optimality theorems assert that even though prices do not enter into the definition, there happens to be an identity between optima and competitive equilibria (under suitable conditions). This relation has been brought into still sharper relief with the modern theory of the core, which also, however, serves to emphasize the special role of large numbers in the theory of perfect competition.

We start with essentially the same model of the production and consumption structure as in Section 2, deleting, of course, all references to prices and to income. However, the analyses of the core have so far made one significant restriction on the relation between producers and consumers. It is assumed that any coalition of households has access to the same set of possible production vectors, which is further assumed to display constant returns to scale. Consider now any feasible allocation. It is said to be blocked by a coalition S (a set of households) if there is another allocation among the members of S feasible for them (using only the assets they collectively started with) which makes each of them better off. Notice that the coalition might consist of all households in the society; for that coalition, blocking reduces to domination in the sense used earlier. A

coalition might also consist of one individual; then he can block an allocation if, with only his own resources and the universally accessible technological knowledge, he can produce a bundle whose utility is higher than that of the bundle allocated to him.

The core, then, consists of all allocations that are not blocked by any coalition. The first theorem generalizes the sufficiency theorem for optimality: Any competitive equilibrium belongs to the core. More interesting is a sort of converse proposition which may be loosely stated as follows: If the hypotheses of Section 2 which insure the existence of equilibrium hold, and if each individual is small compared with the economy, then the allocations in the core are all approximately competitive equilibria. (The words "small" and "approximately" are rigorously interpreted as referring to suitably chosen limiting processes.)

Some interpretive remarks are in order here:

(1) The natural interpretation of the core is that if any sort of bargains are permitted by the rules of the economic game, the allocation finally arrived at should be in the core, since otherwise some coalition would have both the power and the desire to prevent it. Hence, it would follow, very strikingly, that for large numbers of participants, the outcome would be the competitive equilibrium, provided the assumptions of Section 2 were satisfied. Even under non-convexity some scattered results suggest that the same holds approximately (i.e., there may be no core in the precise definition, but there is a set of allocations that can be blocked but only with very small preference on the part of the blocking coalition). Hence, the existence of monopoly must depend on one of three factors: (1) specialized abilities scarce relative to the economy; (2) increasing returns on a scale comparable to that of the economy, or (3)

costs of coalition formation which are relatively low among producers of the same good and high for coalitions involving both consumers and producers.

(2) The assumption that the production possibilities are the same for all coalitions is one that has been used by McKenzie and, with suitable interpretation, is not as drastic as it seems. We can assume that some or all productive processes require as inputs "entrepreneurial skills" or special talents of some kind. The commodity space is enlarged to include these skills, which may be distributed very unequally in the population. Then it can be argued that diminishing returns to scale in the observed variables really results from a combination of constant returns in all variables including entrepreneurial skills and a fixed supply of the latter. Further, different coalitions will really have very different access to production possibilities because of their very different endowments of skills.

4. Uniqueness of Competitive Equilibrium

From this point on, results have been stated only for the case where the excess demand functions are single-valued, as at the beginning of Section 2. It will then be assumed that assumptions (H), (W), and (B) hold; in fact, (C) will be strengthened to require differentiability of the excess demand functions.

Without further assumptions, there is no need that equilibrium be unique, and examples of non-uniqueness have been known since Marshall. The mathematical basis for a fairly general uniqueness theorem has only recently been worked out by Gale and Nikaidô, and the most appropriate economic theorem has not been fully explored. However, one theorem along

these lines can be stated. Suppose there is one commodity for which the excess demand is infinite whenever its price is zero, regardless of the prices of all other commodities. Such a commodity is an eminent candidate for Walras' role of numeraire, and we may choose its price to be one since it cannot be zero (and therefore will be positive) in any equilibrium and, from the homogeneity of excess demand functions, multiplying any equilibrium set of prices by a positive number leads to a new equilibrium (uniqueness of equilibrium is of course defined only up to positive multiples). Call this the n^{th} commodity, and consider the excess demands for commodities $1,\ldots,n-1$ as functions of $p_1,\ldots,p_{n-1}$, with p_n held constant at 1. The Jacobian of this set of functions is defined in mathematics as the matrix with components, $(\partial X_i/\partial p_j)$, where i and j range from 1 to $n-1$. As noted in Section 1, a matrix is termed Hicksian if the determinant of a principal minor is positive when it has an even number of rows, and negative otherwise.

Uniqueness Theorem 1. If the Jacobian of the excess demand functions, omitting a numeraire and holding its price constant, is Hicksian, then the equilibrium is unique.

A special case of this theorem originated in effect with A. Wald. Commodity i will be said to be a gross substitute for commodity j if an increase in p_j, holding all other prices constant, increases X_i. It follows from (H) that if all commodities are gross substitutes, the Jacobian of the excess demand functions, omitting a numeraire, is Hicksian. Then a consequence of Uniqueness Theorem 1 is:

If all commodities are gross substitutes, then equilibrium is unique.

Finally, an entirely different sufficient condition was also stated by Wald:

Uniqueness Theorem 2. If the Weak Axiom of Revealed Preference holds for consumers as a whole, then equilibrium is unique.

5. Stability

The stability problem, as formalized by Samuelson, can be stated as follows: Suppose that an arbitrary (in general, non-equilibrium) set of prices is given, so that there are non-zero excess demands, some positive. It is assumed that prices adjust under the influence of the excess demands, specifically rising when excess demand is positive and falling in the opposite case. This suggests the following dynamic system:

$$dp_i/dt = k_i X_i(p_1, \ldots, p_n) \quad (i = 1, \ldots, n), \tag{1}$$

so that the change in prices is proportional to the excess demand. Notice that allowing for "speeds of adjustment," k_i, which are different from 1 and from each other, is not merely due to a desire for generality but virtually a logical necessity, for a careful dimensional analysis shows that k_i will change with changes in the units of measurement of commodity i. More general (nonlinear) adjustment models have been studied, for example, that dp_i/dt has merely the same sign as X_i.

A variation of this system, which has often been studied, distinguishes one commodity as numeraire, and assumes that its price is held fixed, say at 1.

$$\begin{aligned} dp_i/dt &= k_i X_i(p_1, \ldots, p_n) \quad (i = 1, \ldots, n-1), \\ p_n &= 1. \end{aligned} \tag{2}$$

One difficulty arises in either system when the rules call for a price to become negative, which can happen if, for some i, $X_i(p_1, \ldots, p_n) < 0$

with $p_1 = 0$. Since the excess demand functions are not even defined for negative prices, the rules must be altered. It has become customary to modify (1) and (2) by requiring that a price remain at zero under these conditions; but whether the rules remain consistent is a difficult mathematical question which has been studied only by Uzawa and Morishima, and then only for very special cases.

The systems (1) or (2) are systems of differential equations; their solutions are time paths of prices which are determined by the specification of initial conditions as well as by the system. The stability question is whether or not the resulting time path converges to an equilibrium. Global stability means that convergence occurs for any initial conditions, local stability that the path converges for initial conditions sufficiently close to an equilibrium. However, at the present time the most interesting results are sufficient for global as well as local stability, so we need not distinguish the two.

It should first be noted that neither of the systems is necessarily stable even if all the hypotheses which insure the existence of equilibrium are satisfied; examples have been supplied by Scarf and Gale. The latter's is particularly simple: Suppose there are two individuals and three commodities. There is no production. Individual 1 starts with a supply of good 1 and individual 2 with supplies of goods 2 and 3. Individual 1 has a utility function involving only goods 2 and 3, while individual 2 wishes only good 1. It is easy to see that there is a unique equilibrium. Now suppose that Giffen's paradox holds with regard to good 2 for individual 1 (i.e., a rise in the price of good 2, holding other prices constant, raises individual 1's demand). Then it is possible to show that, for suitably

suitably chosen adjustment speeds, the solution of system (1) or (2) remains away from the equilibrium.

There are three different conditions, any one of which is sufficient for the stability of the system, due to Arrow, H. D. Block, and L. Hurwicz.

Stability Theorem 1. If all commodities are gross substitutes, then systems (1) and (2) are both stable.

Stability Theorem 2. If the market satisfies the Weak Axiom of Revealed Preference, then systems (1) and (2) are both stable.

To state the third theorem we have to introduce the mathematical concept of a matrix with dominant diagonal. A matrix (a_{ij}) is said to have a dominant diagonal if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{for all} \quad i.$$

(The diagonal element is more important than all others in the same row.)

Stability Theorem 3. If the Jacobian of the excess demand functions (excluding a numeraire) has a dominant diagonal, all elements of which are negative, then system (2) is stable.

Whether the Jacobian has a dominant diagonal may depend on the units in which the commodities are measured; Stability Theorem 3 asserts stability if there is any way of choosing the units so as to achieve diagonal dominance.

Stability Theorem 2 can be interpreted as meaning that the transfers of income which take place during the course of the time path produce broadly offsetting results on demand; income effects are not too asymmetrical. Stability Theorem 3 is perhaps closest to Walras' initial concepts; stability holds when the excess demand for a commodity is much more affected by a change in its own price than by any other price change

(holding the price of a numeraire constant).

In all discussion of stability so far it has been implicitly assumed that no transactions take place at non-equilibrium prices, for if they did the excess demand functions would shift. (An alternative interpretation is that all commodities are completely perishable, and utilities and production are independent as between time periods. Then any transactions occurring in one period will have no effect on the next.) This assumption is the classical one of "recontracting" in Edgeworth's terminology, and was made by him and by Walras. The problem was immediately recognized but little analysis took place. Several recent writers, particularly Hahn, Negishi, and Uzawa, have considered it under the rather awkward title of "non-tâtonnement stability." The system, to be complete, has to specify the nature of the transactions; since the system is not at equilibrium, there will have to be rationing of sellers or of buyers. If it is simply assumed that any transactions that do take place cannot change the value of any individual's holdings, then gross substitutability is again a sufficient condition for stability. Under more specific assumptions about transactions, stability can be shown in much wider classes of cases.

6. Comparative Statics

The question raised under this head is: What can be said of the effect of a shift in the excess demand functions on the price system? As might be supposed from the nature of the question, answers can only be given in a limited range of cases. It is supposed that a binary shift has occurred; that is, the excess demand for one commodity, say 1, has decreased at every set of prices, the excess demand for another commodity, say 2, has increased correspondingly (the money value of the increased

demand for 2 exactly equals the decrease in money value of demand for 1 at any given set of prices), and the excess demand functions for all other commodities have remained unchanged. Then the only general result in the literature is the following (Morishima, 1960):

If all commodities are gross substitutes and there has occurred a binary shift in demand from commodity 1 to commodity 2, then all prices of commodities other than 1 rise relative to the price of 1 or do not fall, and no relative price rise is greater than that of commodity 2.

7. Equilibrium Over Time

While the above summarizes the central part of the literature on general economic equilibrium, there is a related conceptual question that deserves brief mention. Consider an economy extending over time, with dated inputs and outputs, and household plans that run into the future. What can be said about the equilibrium of such an economy, and indeed what is meant by the term?

One straightforward answer is that originally due to Hicks. We may simply date all commodity transactions and regard a commodity at one time as being a different commodity. Then the formal model of Section 2 remains, with reinterpretation, and we can still argue that there is an equilibrium over time. Planned supplies and demands are equated in the usual way.

This is a legitimate and indeed important interpretation. Problems of optimality, the core, uniqueness and comparative statics (perhaps to be renamed "comparative dynamics") are restatable with no difficulty. Stability theory faces a more serious challenge since time now enters in two different ways, in the underlying model and in the adjustment process.

If all markets are taken to be futures markets so that all adjustments take place simultaneously, there is no difficulty, but otherwise there is a new range of problems which have been approached only in the most fragmentary way.

However, the simple redating has the important implication of perfect foresight (possibly achieved through having all future economic transactions determined in currently existing futures markets). This seems empirically most unsatisfactory. An alternative is to assume that each individual has expectations about the future that are continuous functions of present observed variables. Then in each period there will be an equilibrium, though the plans for the future made by each individual will not in fact be carried out in general.

From either point of view, considerable interest has attached to another meaning of equilibrium, which we may term <u>stationary equilibrium</u>. The equilibrium over time in the case of perfect foresight defines a set of prices and quantities for each period. The same is true of the succession of short-run equilibria defined by individuals acting under expectations. The question is: Does this time sequence have a stationary or equilibrium value? Is there a set of prices and quantities such that, if they governed in the initial period, would remain equilibrium values for all subsequent periods? Or, in a growing economy, would at least the relative prices and quantities remain constant if the appropriate values held in the initial period? This might be termed the question of existence of stationary equilibrium, frequently termed the <u>balanced growth path</u>.

The stability of stationary equilibrium is a different problem from that of stability in the sense used in Section 5. Suppose we have an

arbitrary initial quantity configuration; will the equilibrium values of the successive periods tend to converge to the stationary equilibrium? The study of these problems has been a major preoccupation of modern growth and capital theory, and will not be enlarged on here.

BIBLIOGRAPHY

Debreu, Gerard (1959), Theory of Value. New York: Wiley and Sons.

Debreu, Gerard (1962), New Concepts and Techniques for Equilibrium Analysis. International Economic Review, 3: 257-73.

Debreu, Gerard, and Herbert Scarf (1963), A Limit Theorem on the Core of an Economy. International Economic Review, 4: 235-46.

Koopmans, Tjalling (ed.) (1951), Activity Analysis of Allocation and Production. New York: Wiley and Sons.

Kuenne, Robert E. (1963), The Theory of General Economic Equilibrium. Princeton: Princeton University Press.

Morishima, Michio (1960), On the Three Hicksian Laws of Comparative Statics. Review of Economic Studies, 27: 195-201.

Negishi, Takashi (1962), The Stability of a Competitive Economy: A Survey Article. Econometrica, 30: 635-69.

Von Neumann, John (1937), Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes. Ergebnisse eines Mathematischen Kolloquiums, 8: 73-83. [English translation (1945): A Model of General Economic Equilibrium. Review of Economic Studies, 13: 1-9.]

Von Neumann, John, and Oskar Morgenstern (1944), Theory of Games and Economic Behavior. Princeton: Princeton University Press.

Wald, Abraham (1936), Über einige Gleichungssysteme der mathematischen Ökonomie. Zeitschrift für Nationalökonomie, 7: 637-70. [English translation (1951): On Some Systems of Equations of Mathematical Economics. Econometrica, 19: 368-403.]

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A brief history of the concept of general economic equilibrium is presented, with special emphasis on the development of the field since 1932. This is followed by a fairly detailed survey of the present known results on the existence of competitive equilibrium and briefer surveys of other topics, specifically the optimality of competitive equilibrium, the concept of the core, uniqueness of competitive equilibrium, stability, comparative statics, and equilibrium over time.

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